

# A GENERALIZATION OF TAKEGOSHI'S RELATIVE VANISHING THEOREM

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**ABSTRACT.** We present a generalization of Takegoshi's relative version of the Grauert-Riemenschneider vanishing theorem. Under some natural assumptions, we extend Takegoshi's vanishing theorem to the case of Nakano semi-positive coherent analytic sheaves on singular complex spaces. We also obtain some new results about proper modifications of torsion-free coherent analytic sheaves.

## 1. Introduction

K. Takegoshi gave in [Tak85] a relative version of the Grauert-Riemenschneider vanishing theorem [GR70]. He proved the vanishing of the higher direct image sheaves of the canonical sheaf under a resolution of singularities. The key ingredient is an  $L^2$ -vanishing theorem for weakly 1-complete Kähler manifolds. These results have many applications, in particular, in the study of singular complex spaces (see e. g. [CS95, CR09]). Therefore, we are interested in generalizations of these so-called relative vanishing theorems.

Let us first describe our setting and explain the notation. Let  $X$  be a locally irreducible complex space of pure dimension  $n$  and  $\pi : M \rightarrow X$  a resolution of singularities (which exists due to H. Hironaka). The direct image sheaf  $\mathcal{K}_X := \pi_* \Omega_M^n$  of the sheaf of holomorphic  $n$ -forms on  $M$  is called the Grauert-Riemenschneider canonical sheaf of  $X$ . It has to be distinguished from the so-called dualizing canonical sheaf  $\omega_X$  of A. Grothendieck.  $\mathcal{K}_X$  is torsion-free and independent of the resolution (see [GR70, § 2.1]). For a smooth plurisubharmonic function<sup>1</sup>  $\Phi$  on  $X$ , we define

$$\sigma(\Phi) := \max_{x \in X_{\text{reg}}} (\text{rk } H(\Phi)_x),$$

where  $H(\Phi)_x$  denotes the complex Hessian of  $\Phi$  at  $x$ . We call  $X$  weakly 1-complete if  $X$  possesses a smooth plurisubharmonic exhaustion function  $\Phi$ . For simplicity, let us assume that  $X$  is connected and non-compact. Since an exhaustion function can not be pluriharmonic (contradiction to the Maximum Principle), we obtain

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<sup>1</sup>A function  $f$  on a complex space  $X$  is called smooth/plurisubharmonic if there exist local embeddings of  $X$  in a complex number space such that  $f$  admits a smooth/plurisubharmonic extension to a neighborhood of  $X$ .

$\sigma(\Phi) > 0$ . We say  $X$  is Kähler if there exists a Kähler form on the regular part  $X_{\text{reg}}$  such that each  $x \in X_{\text{sing}}$  has an open neighborhood  $U = U(x) \subset X$  which can be embedded in  $V \subset \mathbb{C}^{d(x)}$  and a Kähler form  $\eta$  on  $V$  with  $\eta|_{U_{\text{reg}}} = \omega$ .

Let  $\mathcal{S}$  be a coherent analytic sheaf on  $X$ . There exists a duality between the coherent analytic sheaves and the linear (fiber) spaces  $L(\mathcal{S})$  over  $X$  (in the sense of G. Fischer, see [Fis67]). We call  $\mathcal{S}$  Nakano semi-positive if there exists a Hermitian form on  $L(\mathcal{S})$  which is Nakano semi-negative on the set where  $L(\mathcal{S})$  is a vector bundle (for more details, see Def. 3.1 or [GR70, § 1.2]). For a holomorphic map  $\pi : Y \rightarrow X$ , we call  $\pi^T \mathcal{S} := \pi^* \mathcal{S} / \mathcal{T}(\pi^* \mathcal{S})$  the torsion-free preimage sheaf of  $\mathcal{S}$ . Here,  $\mathcal{T}(\pi^* \mathcal{S})$  denotes the torsion (sub-) sheaf of  $\pi^* \mathcal{S}$ .

**Def. 1.1.** We say that a coherent analytic sheaf  $\mathcal{S}$  on  $X$  fulfills condition (+) if there exist a projective morphism  $\pi : \tilde{X} \rightarrow X$  with locally free  $\pi^T \mathcal{S}$  and a semi-positive invertible coherent analytic sheaf  $\mathcal{L}$  on  $\tilde{X}$  such that  $\pi^T \mathcal{K}_X \cong \mathcal{L} \otimes \mathcal{K}_{\tilde{X}}$ . If these exist on all relative compact weakly 1-complete open subsets of  $X$ , we say that  $\mathcal{S}$  satisfies  $(+)_{\text{loc}}$ .

We can now state our first main result:

**Theorem I.** *Let  $X$  be a weakly 1-complete connected normal<sup>2</sup> Kähler space of dimension  $n$  with locally free canonical sheaf (e.g.  $X$  is Gorenstein and has canonical singularities), let  $\Phi$  be a smooth plurisubharmonic exhaustion function of  $X$ , and let  $\mathcal{S}$  be a Nakano semi-positive torsion-free sheaf on  $X$  with (+) such that  $L(\mathcal{S})$  is normal. Then, for each  $q > n - \sigma(\Phi)$ ,*

$$H^q(X, \mathcal{S} \otimes \mathcal{K}_X) = 0$$

*if  $H^q(X, \mathcal{S} \otimes \mathcal{K}_X)$  and  $H^{q+1}(X, \mathcal{S} \otimes \mathcal{K}_X)$  are Hausdorff.*

This is a generalization of Thm. 2.1 in [Tak85]. If  $X$  is holomorphically convex, then the Hausdorff assumption is always satisfied, [Pri71, Lem. II.1]. We remark also that the Kähler structure of  $X$  and the Nakano semi-positivity of  $\mathcal{S}$  are needed only on relative compact subsets of  $X$  (cf. Theorem 2.9).

The proof of Theorem I uses that the isomorphism induced by the Leray spectral sequence is already topological (see Theorem 2.14). Actually, this is easy to prove although, to the knowledge of the author, it has not been observed in the literature (yet, [Pri71, Lem. II.1] and its proof are interesting in this context).

Obviously, locally free sheaves satisfy (+) and their associated linear spaces are normal. In this case, we get the vanishing theorem for arbitrary irreducible complex spaces (see Theorem 2.17).

In Section 2.1 and 2.2, we first prove the result in the regular case (for vector bundles on manifolds), following mainly the lines of Takegoshi's original. The key results (positivity statements and an a-priori-estimate) are achieved by  $L^2$ -methods elaborated by J.-P. Demailly in [Dem02]. By use of the projection

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<sup>2</sup>Actually, it is just needed to assume that  $X$  is locally irreducible: the assumption that  $\mathcal{K}_X$  is locally free implies that  $X$  is normal (see Theorem 4.7).

formula (see (2.16)), we obtain the generalization to singular complex spaces (see Section 2.3).

Since there is no projection formula for non-locally-free sheaves, the situation is much more complicated for such sheaves. We then need to assume that  $L(\mathcal{S})$  is normal, and the additional positivity property (+). If the linear space associated to  $\mathcal{S} \otimes \mathcal{K}_X$  is normal, J. Ruppenthal and the author proved that there is a canonical isomorphism

$$\mathcal{S} \otimes \mathcal{K}_X \cong \pi_*(\pi^T \mathcal{S} \otimes \pi^T \mathcal{K}_X)$$

for all modifications  $\pi$  of  $X$  (see Thm. 1.4 in [RS13]). Here, one needs a suitable connection between  $\pi^T \mathcal{K}_X$  and  $\mathcal{K}_{\tilde{X}}$ . If (+) holds, then we obtain  $\mathcal{S} \otimes \mathcal{K}_X$  as the direct image of a Nakano semi-positive locally free sheaf tensored with the canonical sheaf. Using the Leray spectral sequence, we can now prove Theorem I (see Section 3.1). The last step was inspired by [GR70] of H. Grauert and O. Riemenschneider. Analogously, one can prove the following corollary of Satz 2.1 in [GR70]:

**Corollary 1.2.** *Let  $X$  be a compact normal Moishezon space with locally free canonical sheaf, and let  $\mathcal{S}$  be a torsion-free quasi-positive sheaf with (+) such that  $L(\mathcal{S})$  is normal. Then, for each  $q > 0$ ,*

$$H^q(X, \mathcal{S} \otimes \mathcal{K}_X) = 0.$$

Let us add a few words of how to verify that  $\mathcal{S}$  satisfies (+). H. Rossi proved that there exists a projective morphism  $\varphi = \varphi_{\mathcal{S}} : X_{\mathcal{S}} \rightarrow X$  such that  $\varphi^T \mathcal{S}$  is locally free (Thm. 3.5 in [Ros68]). In [Rie71, § 2], O. Riemenschneider showed that this construction has universal properties and called it the monoidal transformation of  $X$  with respect to  $\mathcal{S}$ . Moreover, we have (see Thm. 8.1 in [RS13]) the following useful fact: For any resolution  $\pi : M \rightarrow X$  of singularities (such that  $\pi^T \mathcal{K}_X$  is locally free), there exists an effective Cartier divisor  $D \geq 0$  (with support on the exceptional set of the resolution) such that

$$\pi^T \mathcal{K}_X \cong \mathcal{K}_M \otimes \mathcal{O}_M(-D).$$

Hence, to hold the property (+), it is just needed that  $\mathcal{O}(-D)$  is semi-positive. In Section 5, we give an example where we see the assumption (+) holds for a non-locally-free sheaf. More precisely, we consider a semi-positive (reduced) ideal sheaf  $\mathcal{J}$  on a weakly 1-complete manifold given by a submanifold and prove that  $\mathcal{J}$  satisfies (+). This is obtained by the semi-positivity of  $\mathcal{J}$  itself, which is an indication for a link between (Nakano) semi-positivity of a sheaf and (+). Using Theorem I, we obtain a vanishing theorem for globally defined submanifolds (see Corollary 5.4).

In Section 4, we show that the normality assumption on  $L(\mathcal{S})$  is necessary for a generalization of Takegoshi's vanishing theorem that makes use of the monoidal transformation (see Remark 4.6). Let us comment a bit on the techniques used here as they are of independent interest.

First, we show that the direct image of the torsion-free preimage of a suitable sheaf  $\mathcal{S}$  of rank 1 under the monoidal transformation with respect to  $\mathcal{S}$  is canonically isomorphic to  $\mathcal{S}$  (see Theorem 4.1). Second, we prove that the torsion-free preimage of the direct image of an arbitrary torsion-free coherent analytic sheaf  $\mathcal{F}$  with respect to a 1:1 modification is canonically isomorphic to  $\mathcal{F}$  (see Theorem 4.5). Third, we show that a locally free sheaf on a non-normal complex space  $X$  can not be the direct image of a sheaf on a normal modification of  $X$  (cf. Theorem 4.7). In particular, the Grauert-Riemenschneider canonical sheaf  $\mathcal{K}_X$  can not be locally free on a non-normal space  $X$ .

The following result (a generalization of Thm. I in [Tak85]) is a conclusion of Theorem I proven in Section 3.2; the presented proof is derived from Takegoshi's.

**Theorem II.** *Let  $X$  be a normal complex space with locally free canonical sheaf which is bimeromorphic to a Kähler space, let  $f : X \rightarrow Z$  be a proper surjective holomorphic map onto a complex space  $Z$ , and let  $\mathcal{S}$  be a semi-globally<sup>3</sup> Nakano semi-positive torsion-free sheaf on  $X$  satisfying  $(+)_\text{loc}$  such that  $L(\mathcal{S})$  is normal. Then the higher direct images of  $\mathcal{S} \otimes \mathcal{K}_X$  under  $f$  vanish for all  $q > \dim X - \dim Z$ :*

$$f_{(q)}(\mathcal{S} \otimes \mathcal{K}_X) = 0.$$

In Section 6, we finally study coherent analytic sheaves with torsion. We prove a generalization of Theorem I for  $q$  strictly larger than the dimension of the support of the associated torsion sheaf. For smaller  $q$ , we give a counterexample.

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## 2. Vanishing theorems for vector bundles

In this section, we generalize Takegoshi's vanishing theorems [Tak85] to Nakano semi-positive vector bundles.

**2.1. A-priori-estimates for the  $\bar{\partial}$ -operator.** Let  $M$  be a weakly 1-complete Kähler manifold of dimension  $n$ ,  $\omega$  the Kähler form on  $M$ , and let  $\Phi$  be a pluri-subharmonic smooth exhaustion function of  $M$ . Let  $\lambda : \mathbb{R} \rightarrow \mathbb{R}$  be an increasing convex smooth function with  $\lambda(t) = 0$  for  $t \leq 0$  and  $\int \sqrt{\lambda''(t)} dt = +\infty$ . By replacing, firstly,  $\Phi$  by  $\lambda \circ \exp \circ \Phi$  and, secondly,  $\omega$  by  $\omega + i\partial\bar{\partial}\Phi$ , we can assume that  $\Phi > 0$  and that the Kähler metric associated to  $\omega$  is complete (see Prop. 12.10 in [Dem02]).

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<sup>3</sup> on relative compact weakly 1-complete sets

**Def. 2.1.** Let  $E \rightarrow M$  be a holomorphic vector bundle of rank  $r$  with Hermitian metric  $\langle \cdot, \cdot \rangle_E$ . We call a tensor  $u \in T_x M \otimes E_x$  of rank  $m$  if  $m \geq 0$  is the smallest integer such that  $u$  is the sum  $\sum_{j=1}^m \xi_j \otimes s_j$  of  $m$  pure/simple tensors. A Hermitian form  $H$  on  $T_x M \otimes E_x$  is called  $m$ -(semi-)positive if  $H(u, u) > 0$  (or  $\geq 0$  resp.) for every tensor  $u \in T_x M \otimes E_x \setminus \{0\}$  of rank  $\leq m$ . We say  $E$  is  $m$ -(semi-)positive if the Hermitian form associated to the Chern curvature  $i\Theta(E)$  is  $m$ -(semi-)positive in each point  $x \in M$  and write  $E >_m 0$  (or  $E \geq_m 0$  resp.). We call  $E$  *Griffiths (semi-) positive* if  $E$  is 1-(semi-)positive, and *Nakano (semi-) positive* if  $E$  is  $\min\{n, r\}$ -(semi-)positive, i. e.,  $i\Theta(E)$  is (semi-)positive in the classical sense.

As an immediate consequence of the definition, we get:

**Lemma 2.2.** *Let  $E$  be an  $m$ -semi-positive holomorphic vector bundle on  $M$  of rank  $r$ . If  $t \geq 1$  and  $m \geq \min\{n-t+1, r\}$ , then the Hermitian operator  $i\Theta(E) \wedge \Lambda$  is semi-positive definite on  $\Lambda^{n,t} T^* M \otimes E$  and, particularly,*

$$\langle i\Theta(E) \wedge \Lambda w, w \rangle_{\omega, E} \geq 0 \quad \forall w \in \mathcal{D}_{n,t}(M, E).$$

Here,  $\Lambda$  denotes the formal adjoint of the operator  $\omega \wedge \cdot$ , and  $\langle \cdot, \cdot \rangle_{\omega, E}$  is the metric on  $\Lambda^{s,t} T^* M \otimes E$  given by  $\omega$  and  $\langle \cdot, \cdot \rangle_E$ .

*Proof:* Follows directly from the proof of Lemma VII.7.2 in [Dem12].  $\square$

Let  $dV$  be the volume form on  $M$  given by  $\omega$ . For a smooth function  $\Phi : M \rightarrow \mathbb{R}$ , we define the weighted product of  $u$  and  $v$  in  $\mathcal{D}_{s,t}(M, E)$  by

$$(u, v)_\Phi := \int_M \langle u, v \rangle_{\omega, E} \cdot e^{-\Phi} dV$$

and set  $\|u\|_\Phi := \sqrt{(u, u)_\Phi}$ . Let  $\vartheta_\Phi$  denote the formal adjoint of  $\bar{\partial}$  with respect to  $(\cdot, \cdot)_\Phi$ . We obtain the following a-priori-estimate for the  $\bar{\partial}$ -operator:

**Lemma 2.3.** *For all  $t \geq 1$  and  $w \in \mathcal{D}_{n,t}(M, E)$ :*

$$(i(\Theta(E) + \partial\bar{\partial}\Phi) \wedge \Lambda w, w)_\Phi \leq \|\bar{\partial}w\|_\Phi^2 + \|\vartheta_\Phi w\|_\Phi^2.$$

*Proof:* Let  $L$  denote the trivial line bundle  $M \times \mathbb{C}$  with the Hermitian metric  $\langle \cdot, \cdot \rangle_L := \langle \cdot, \cdot \rangle_{\mathbb{C}} e^{-\Phi}$ . Then  $i\Theta(L) = i\partial\bar{\partial}\Phi$ . Let  $F := E \otimes L$  denote the vector bundle with the metric  $\langle \cdot, \cdot \rangle_F$  induced by  $\langle \cdot, \cdot \rangle_E$  and  $\langle \cdot, \cdot \rangle_L$ . We can assume  $\mathcal{D}_{s,t}(M, E) = \mathcal{D}_{s,t}(M, F)$  by identifying  $u \otimes g = (g \wedge u) \otimes 1$  with  $g \wedge u$ . Thus, we get  $\langle \cdot, \cdot \rangle_F = \langle \cdot, \cdot \rangle_E \cdot e^{-\Phi}$ . With  $\Theta(F) = \Theta(E) \otimes \text{id}_L + \text{id}_E \otimes \Theta(L)$  (cf. [Dem12, § V.4]), we conclude for  $u \in \mathcal{D}_{s,t}(M, E)$ :

$$\begin{aligned} \Theta(F) \wedge u &= \Theta(F) \wedge (u \otimes 1) = (\Theta(E) \wedge u) \otimes 1 + u \otimes \Theta(L) \\ &= (\Theta(E) + \Theta(L)) \wedge u \otimes 1 = (\Theta(E) + \partial\bar{\partial}\Phi) \wedge u \end{aligned}$$

Let  $\Delta_F^{(i)} := D_F^{(i)} D_F^{(i)*} + D_F^{(i)*} D_F^{(i)}$  denote the Laplace-Beltrami operators, where  $D_F = D'_F + D''_F$  is the Chern connection with respect to the metric  $\langle \cdot, \cdot \rangle_E e^{-\Phi}$ . Note that  $D''_F = \bar{\partial}$  and  $(D''_F)^* = \vartheta_\Phi$ . Recall the Bochner-Kodaira-Nakano identity (see e. g. [Dem02, Sect. 13.2]):

$$\Delta_F'' = \Delta_F' + [i\Theta(F), \Lambda].$$

Integration by parts yields

$$(\Delta_F^{(i)} u, u)_\Phi = \|D_F^{(i)} u\|_\Phi^2 + \|(D_F^{(i)})^* u\|_\Phi^2 \quad \forall u \in \mathcal{D}_{s,t}(M, E).$$

Altogether, we conclude

$$\begin{aligned} \|\bar{\partial} w\|_\Phi^2 + \|\vartheta_\Phi w\|_\Phi^2 &= (\Delta_F'' w, w)_\Phi = \|D_F' w\|_\Phi^2 + \|(D_F')^* w\|_\Phi^2 + ([i\Theta(F), \Lambda] w, w)_\Phi \\ &\geq \int_M \langle i\Theta(F) \wedge \Lambda w, w \rangle_{\omega, E} e^{-\Phi} dV = (i(\Theta(E) + \partial\bar{\partial}\Phi) \wedge \Lambda w, w)_\Phi \end{aligned}$$

for all  $w \in \mathcal{D}_{n,t}(M, E)$ .  $\square$

We will combine the a-priori-estimate with the following positivity statement:

**Lemma 2.4.** *Choose some integers  $q, t$  with  $t \geq q \geq 1$ . Then there is a non-negative (bounded) continuous function  $\delta$  on  $M$  such that  $\delta(x) > 0$  for all points  $x \in M$  satisfying  $\text{rk } H(\Phi)_x > n - q$ , and*

$$\delta \cdot \langle w, w \rangle_{\omega, E} \leq \langle i\partial\bar{\partial}\Phi \wedge \Lambda w, w \rangle_{\omega, E} \quad \forall w \in \mathcal{D}_{n,t}(M, E).$$

*Proof:* Fix a point  $x$  in  $M$ . There is a base  $\{\frac{\partial}{\partial z_j}\}_{j=1}^n$  of  $T_x^\mathbb{C} M$  such that

$$\omega(x) = i \sum_{j=1}^n dz_j \wedge d\bar{z}_j \quad \text{and} \quad i\partial\bar{\partial}\Phi(x) = i \sum_{j=1}^n \delta_j dz_j \wedge d\bar{z}_j$$

with  $\delta_1, \dots, \delta_n \in \mathbb{R}$ . For  $u = \sum u_{JK} dz_J \wedge d\bar{z}_K \in \mathcal{D}_{s,t}(M)$ , we get (see e. g. Prop. 6.8 in [Dem02, Sect. 6.B])

$$[i\partial\bar{\partial}\Phi, \Lambda] u(x) = \sum_{J,K} \left( \sum_{j \in J} \delta_j + \sum_{j \in K} \delta_j - \sum_{j=1}^n \delta_j \right) u_{J,K}(x) dz_J \wedge d\bar{z}_K. \quad (2.5)$$

Let us define  $\delta_{J,K} := \sum_{j \in J} \delta_j + \sum_{j \in K} \delta_j - \sum_{j=1}^n \delta_j$ . Choose a frame  $\{e_1, \dots, e_r\}$  of  $E$  on a small open neighborhood of  $x$ . Let  $(h_{\lambda\mu})$  denote the Hermitian matrix associated to the Hermitian metric  $\langle \cdot, \cdot \rangle_E$  on  $E$  such that  $\langle u, v \rangle_{\omega, E} = \sum_{\lambda, \mu=1}^r h_{\lambda\mu} \langle u_\lambda, v_\mu \rangle_\omega$  for  $u = \sum u_\lambda \otimes e_\lambda$ ,  $v = \sum v_\lambda \otimes e_\lambda \in \mathcal{D}_{s,t}(M, E)$ . For  $u = \sum u_{J,K,\lambda} dz_J \wedge d\bar{z}_K \otimes e_\lambda \in \mathcal{D}_{s,t}(M, E)$ , we obtain (in the point  $x$ ):

$$\begin{aligned} \langle [i\partial\bar{\partial}\Phi, \Lambda] u, u \rangle_{\omega, E} &= \sum_{\lambda, \mu} h_{\lambda\mu} \langle [i\partial\bar{\partial}\Phi, \Lambda] u_\lambda, u_\mu \rangle_\omega \\ &\stackrel{(2.5)}{=} \sum_{\lambda, \mu} h_{\lambda\mu} \left\langle \sum_{J,K} \delta_{J,K} u_{J,K,\lambda} dz_J \wedge d\bar{z}_K, u_\mu \right\rangle_\omega \\ &= \sum_{J,K,\lambda,\mu} \delta_{J,K} h_{\lambda\mu} \langle u_{J,K,\lambda} dz_J \wedge d\bar{z}_K, u_{J,K,\mu} dz_J \wedge d\bar{z}_K \rangle_\omega \\ &= \sum_{J,K} \delta_{J,K} \langle u_{J,K}, u_{J,K} \rangle_E = \sum_{J,K} \delta_{J,K} |u_{J,K}|_E^2, \end{aligned}$$

where  $u_\lambda := \sum_{J,K} u_{J,K,\lambda} dz_J \wedge d\bar{z}_K$  and  $u_{J,K} := \sum_\lambda u_{J,K,\lambda} \otimes e_\lambda$ . Now, we define  $\delta(x)$  as the maximum of the  $q$  smallest  $\delta_j$ . Moreover, we get  $\delta_{\{1, \dots, n\}, K} = \sum_{j \in K} \delta_j \geq \delta(x)$  if  $|K| \geq q$ . Hence,

$$\langle [i\partial\bar{\partial}\Phi, \Lambda] w, w \rangle_{\omega, E}(x) \geq \delta(x) \sum_K |w_K|_E^2(x) = \delta(x) \langle w, w \rangle_{\omega, E}(x)$$

for all  $w = \sum w_K dz_{\{1,\dots,n\}} \wedge d\bar{z}_K \in \mathcal{D}_{n,t}(M, E)$  and  $t \geq q$ . Obviously,  $\delta$  is continuous in  $x$  and positive where  $\text{rk } H(\Phi)_x > n - q$ .  $\square$

The three lemmata together imply, assuming  $E$  is an  $m$ -semi-positive holomorphic vector bundle of rank  $r$  on  $M$ ,

$$(\delta \cdot w, w)_\Phi \leq \|\bar{\partial}w\|_\Phi^2 + \|\vartheta_\Phi w\|_\Phi^2 \quad (2.6)$$

for all  $w \in \mathcal{D}_{n,t}(M, E)$  and  $t \geq 1$  if  $m \geq \min\{n, r\}$ , i. e.,  $E$  is Nakano semi-positive, or  $t \geq \min\{n - m + 1, r\}$  else.

**2.2.  $L^2$ -vanishing theorems.** Keeping the setting and notations of the former subsection,  $L_{s,t}^2(M, E; \Phi)$  denotes the square integrable forms  $u$  with respect to the norm  $\|\cdot\|_\Phi$ , and  $\bar{\partial} = \bar{\partial}_w$  denotes the weak/maximal extension of  $\bar{\partial} : \mathcal{D}_{s,t}(M, E) \rightarrow \mathcal{D}_{s,t+1}(M, E)$ , i. e.,  $\bar{\partial}$  in the sense of distributions. Since the metric on  $M$  is complete, we obtain that the weak extension and the strong/minimal extension (given by the closure of the graph) of the  $\bar{\partial}$ -operator coincide. Therefore,  $\vartheta_\Phi = \vartheta_{\Phi,w}$  in the sense of distributions coincides with the Hilbert-space adjoint of  $\bar{\partial} = \bar{\partial}_w$ . Let  $\bar{*}_\Phi : L_{s,t}^2(M, E; \Phi) \rightarrow L_{n-s,n-t}^2(M, E^*; -\Phi)$  be the conjugated Hodge-\* operator defined by

$$\langle u, v \rangle_{\omega, E} \cdot e^{-\Phi} = u \wedge \bar{*}_\Phi v \quad \text{for } u, v \in L_{s,t}^2(M, E; \Phi).$$

**Theorem 2.7.** *Let  $M$  be a complete Kähler manifold of dimension  $n$ , let  $\Phi$  be a smooth plurisubharmonic exhaustion function of  $M$ , and let  $E \rightarrow M$  be a Nakano semi-positive holomorphic vector bundle. Then the following groups of harmonic forms are zero for  $q > n - \sigma(\Phi)$ <sup>4</sup>:*

$$\begin{aligned} \mathcal{H}_{L^2(\Phi)}^{n,q}(M, E) &:= \{u \in L_{n,q}^2(M, E; \Phi) : \bar{\partial}u = 0, \vartheta_\Phi u = 0\} = 0 \quad \text{and} \\ \mathcal{H}_{L^2(-\Phi)}^{0,n-q}(M, E^*) &= 0. \end{aligned}$$

*Proof:* Using that the given metric is complete, it is well-known that  $\bar{*}_\Phi$  induces the  $L^2$ -duality  $\mathcal{H}_{L^2(\Phi)}^{n,q}(M, E) \cong \mathcal{H}_{L^2(-\Phi)}^{0,n-q}(M, E^*)$ . Let  $h$  be a harmonic form in  $\mathcal{H}_{L^2(\Phi)}^{n,q}(M, E)$ . Then  $h$  is in the kernel of the elliptic weighted Laplace operator  $\square_\Phi := \Delta_\Phi''$  so that  $h$  is smooth, i. e.,  $h \in L_{n,q}^2(M, E; \Phi) \cap \mathcal{E}_{n,q}(M, E)$ . As the metric is complete,  $\mathcal{D}_{n,q}(M, E)$  is dense in  $\text{dom } \bar{\partial} \cap \text{dom } \vartheta_\Phi$  with respect to the graph norm  $u \mapsto \|u\|_\Phi + \|\bar{\partial}u\|_\Phi + \|\vartheta_\Phi u\|_\Phi$ . Hence, there is a sequence  $\{h_k\}$  in  $\mathcal{D}_{n,q}(M, E)$  with  $h_k \rightarrow h$ ,  $\bar{\partial}h_k \rightarrow \bar{\partial}h = 0$  and  $\vartheta_\Phi h_k \rightarrow \vartheta_\Phi h = 0$  in  $L_{n,q}^2(M, E; \Phi)$ . With (2.6), we obtain

$$\begin{aligned} (\delta \cdot h, h)_\Phi &= (\delta \cdot (h - h_k), h)_\Phi + (\delta \cdot h_k, h - h_k)_\Phi + (\delta \cdot h_k, h_k)_\Phi \\ &\leq \|h - h_k\|_\Phi \cdot \|h\|_\Phi + \|h_k\|_\Phi \cdot \|h - h_k\|_\Phi + \|\bar{\partial}h_k\|_\Phi^2 + \|\vartheta_\Phi h_k\|_\Phi^2 \\ &\rightarrow 0 \text{ if } k \rightarrow \infty. \end{aligned}$$

Therefore, the harmonic form  $h$  vanishes on the open set  $\{x \in M : \delta(x) > 0\} \supset \{x \in M : \text{rk } H(\Phi)_x > n - q\}$ . But  $\{x \in M : \text{rk } H(\Phi)_x > n - q\}$  is not empty for

<sup>4</sup> Recall  $\sigma(\Phi) := \max_{x \in M} (\text{rk } H(\Phi)_x)$  where  $H(\Phi)_x$  denotes the complex Hessian of  $\Phi$  at  $x$ .

$q > n - \sigma(\Phi)$ . So, the Unique Continuation Theorem (see [Aro57]) implies that  $h$  vanishes on  $M$ .  $\square$

Before proving Theorem I for vector bundles on complex manifolds, let us recall the following criterion of A. Andreotti & E. Vesentini (see [AV63, Prop. 41 & Lem. 12]). For  $v \in \mathcal{D}_{n-s, n-t}(M, E^*)$ , we define the distribution  $T_v : \mathcal{E}_{s,t}(M, E) \rightarrow \mathbb{C}$  by

$$T_v u := \int_M v \wedge u.$$

**Theorem 2.8** (Andreotti-Vesentini). *Let  $M$  be a complex manifold of dimension  $n$ , let  $E \rightarrow M$  be a holomorphic vector bundle on  $M$ , let  $\bar{\partial} : \mathcal{E}_{s,t}(M, E) \rightarrow \mathcal{E}_{s,t+1}(M, E)$  be a topological homomorphism, and let  $v \in \mathcal{D}_{n-s, n-t}(M, E^*)$  be a  $\bar{\partial}$ -closed form with values in  $E^*$ . Then the equation  $\bar{\partial}w = v$  has a solution  $w \in \mathcal{D}_{n-s, n-t-1}(M, E^*)$  if and only if  $T_v u = 0$  for all  $\bar{\partial}$ -closed  $u \in \mathcal{E}_{s,t}(M, E)$ .*

**Theorem 2.9.** *Let  $M$  be a weakly 1-complete complex manifold of dimension  $n$ , and let  $E \rightarrow M$  be a Nakano semi-positive holomorphic vector bundle on  $M$ . Assume that  $M$  admits a smooth plurisubharmonic exhaustion function  $\Phi$  such that the sublevel sets  $M_l := \{x \in M : \Phi(x) < l\} \Subset M$  are Kähler,  $l \in \mathbb{N}$  (i. e.,  $M$  is Kähler on relative compact sets). Then for all  $q > n - \sigma(\Phi)$ :*

$$H_{\text{cpt}}^{n-q}(M, \mathcal{O}_{E^*}) \cong H_{\text{cpt}}^{0, n-q}(M, E^*) = 0$$

*if and only if  $H^{q+1}(M, \Omega_E^n)$  is Hausdorff. In this case, the following is equivalent:*

- (1)  $H^q(M, \Omega_E^n)$  is Hausdorff.
- (2)  $H_{\text{cpt}}^{n-q+1}(M, \mathcal{O}_{E^*})$  is Hausdorff.
- (3)  $H^{n,q}(M, E) \cong H^q(M, \Omega_E^n) = 0$ .

*If  $M$  is holomorphically convex, then all mentioned cohomology spaces vanish.*

*Proof:* The Dolbeault isomorphism theorem yields

$$H^t(M, \Omega_E^s) \cong H^{s,t}(M, E) := \{u \in \mathcal{E}_{s,t}(M, E) : \bar{\partial}u = 0\} / \bar{\partial}\mathcal{E}_{s,t-1}(M, E)$$

and

$$H_{\text{cpt}}^t(M, \Omega_{E^*}^s) \cong H_{\text{cpt}}^{s,t}(M, E^*) := \{u \in \mathcal{D}_{s,t}(M, E^*) : \bar{\partial}u = 0\} / \bar{\partial}\mathcal{D}_{s,t-1}(M, E^*).$$

Let us first prove the implication  $H^{q+1}(M, \Omega_E^n)$  Hausdorff  $\Rightarrow H_{\text{cpt}}^{0, n-q}(M, E^*) = 0$ : Since  $H^{q+1}(M, \Omega_E^n)$  is Hausdorff,  $\bar{\partial} : \mathcal{E}_{n,q}(M, E) \rightarrow \mathcal{E}_{n,q+1}(M, E)$  is a topological homomorphism on Fréchet spaces (see e. g. Prop. 6 of [Ser55]). Hence, the assumptions of Theorem 2.8 are satisfied for  $(s, t) = (n, q)$  and we can use it to show that  $H_{\text{cpt}}^{0, n-q}(M, E^*) = 0$ :

So, let  $v \in \mathcal{D}_{0, n-q}(M, E^*)$  be  $\bar{\partial}$ -closed. We have to show that  $v$  is  $\bar{\partial}$ -exact. Choose an  $l \in \mathbb{N}$  such that  $\text{supp } v \subset M_l$  and  $\sigma(\Phi|_{M_l}) = \sigma(\Phi)$ , i. e.,  $M_l$  contains a point  $x$  where  $\text{rk } H(\Phi)_x$  is maximal. By Theorem 2.8, it suffices to show that  $T_v u = 0$  for all  $u \in \mathcal{E}_{n,q}(M_l, E)$  with  $\bar{\partial}u = 0$ . Fix such a  $u$ .



Choosing an appropriate smooth increasing convex function  $\lambda : (-\infty, l) \rightarrow \mathbb{R}^+$  with  $\lim_{t \rightarrow l} \lambda(t) = \infty$ , we get (i) a smooth plurisubharmonic exhaustion function  $\Psi := \lambda \circ \Phi$  of  $M_l$ , (ii) the Kähler metric given by  $\omega$  is complete on  $M_l$  (replace  $\omega$  by  $\omega + i\partial\bar{\partial}\Psi$ ), and (iii)  $u \in L^2_{n,q}(M_l, E; \Psi) \cap \mathcal{E}_{n,q}(M_l, E)$ . Thus,  $g := (-1)^{n+q} \bar{*}_\Psi u \in L^2_{0,n-q}(M_l, E^*; -\Psi) \cap \mathcal{E}_{0,n-q}(M_l, E^*)$  and  $\vartheta_{-\Psi} g = 0$ . Since  $\ker \vartheta_{-\Psi} \cap \ker \bar{\partial} = \mathcal{H}^{0,n-q}_{L^2(-\Psi)}(M_l, E^*) = 0$  (see Theorem 2.7), we get  $g \in \ker \vartheta_{-\Psi} = (\ker \bar{\partial})^\perp = \overline{\mathcal{R}(\vartheta_{-\Psi})}$ . Hence, there is a sequence  $\{f_k\}$  in  $\mathcal{D}_{0,n-q+1}(M_l, E^*)$  with

$$\|g - \vartheta_{-\Psi} f_k\|_{-\Psi} \rightarrow 0 \text{ if } k \rightarrow \infty.$$

Finally, we infer

$$\begin{aligned} T_v u &= \int_{M_l} v \wedge u = \int_{M_l} v \wedge \bar{*}_{-\Psi} g = (v, g)_{-\Psi} \\ &= \lim_{k \rightarrow \infty} (v, \vartheta_{-\Psi} f_k)_{-\Psi} = \lim_{k \rightarrow \infty} (\bar{\partial} v, f_k)_{-\Psi} \stackrel{\bar{\partial} v = 0}{=} 0. \end{aligned}$$

This shows that indeed  $H_{\text{cpt}}^{n-q}(M, \mathcal{O}_{E^*}) \cong H_{\text{cpt}}^{0,n-q}(M, E^*) = 0$ .

To prove the other implications, we use the following result of H. Laufer (see Thm. 3.2 in [Lau67]<sup>5</sup>): There exist linear topological spaces  $R = R^{q+1}(M, \Omega_E^n)$  and  $R_{\text{cpt}} = R_{\text{cpt}}^{n-q+1}(M, \mathcal{O}_{E^*})$  such that

$$H^q(M, \Omega_E^n) \cong H_{\text{cpt}}^{n-q}(M, \mathcal{O}_{E^*})^* \oplus R_{\text{cpt}}, \quad (2.10)$$

$$H_{\text{cpt}}^{n-q}(M, \mathcal{O}_{E^*}) \cong H^q(M, \Omega_E^n)^* \oplus R, \quad (2.11)$$

$$R = 0 \Leftrightarrow H^{q+1}(M, \Omega_E^n) \text{ is Hausdorff} \quad (2.12)$$

$$R_{\text{cpt}} = 0 \Leftrightarrow H_{\text{cpt}}^{n-q+1}(M, \mathcal{O}_{E^*}) \text{ is Hausdorff}. \quad (2.13)$$

Using (2.11),  $H_{\text{cpt}}^{n-q}(M, \mathcal{O}_{E^*}) = 0$  implies  $R = 0$ , i. e.,  $H^{q+1}(M, \Omega_E^n)$  is Hausdorff. In this case, (2.11) implies

$$H^q(M, \Omega_E^n)^* \cong H_{\text{cpt}}^{n-q}(M, \mathcal{O}_{E^*}) = 0.$$

If  $H^q(M, \Omega_E^n)$  is Hausdorff, then  $H^q(M, \Omega_E^n)$  has to vanish, i. e., (1)  $\Rightarrow$  (3). The converse (3)  $\Rightarrow$  (1) is trivial. Finally, (2.10) and (2.13) give us (2)  $\Leftrightarrow$  (3) immediately. Actually, the equivalence (1)  $\Leftrightarrow$  (2) can directly be proven with functional analysis tools.

It is well known that the sheaf-cohomologies for coherent analytic sheaves on holomorphically convex manifolds are Hausdorff (see Lem. II.1 in [Pri71]).  $\square$

**2.3. Irreducible complex spaces.** In this subsection, we will prove Takegoshi's vanishing theorem for locally free sheaves on irreducible complex spaces. For this, we indicate how a vanishing theorem as Theorem 2.9 yields vanishing of some higher direct image sheaves. We will need this observation later in the proof of Theorem I and II, as well.

<sup>5</sup>The result of H. Laufer is a generalization of the Serre duality (see [Ser55, Thm. 2]). He treated the case where  $\bar{\partial}$  is not necessarily a topological homomorphism.

**Theorem 2.14.** *Let  $X$  be a complex space of pure dimension  $n$ , and let  $\mathcal{F}$  be a coherent analytic sheaf on  $X$  such that the following property is satisfied: For every relative compact holomorphically convex<sup>6</sup>  $U \subset X$  with a smooth plurisubharmonic exhaustion function  $\Phi$ , we have  $H^r(U, \mathcal{F}) = 0$  for all  $r > n - \sigma(\Phi)$ . Further, we assume there is a proper surjective holomorphic map  $f : X \rightarrow Z$  to a complex space  $Z$ . For each  $r > n - \dim Z$ , we get*

$$f_{(r)}(\mathcal{F}) = 0.$$

*If  $\dim Z = n$ , the isomorphism*

$$H^q(X, \mathcal{F}) \cong H^q(Z, f_*\mathcal{F})$$

*induced by the Leray spectral sequence is topological for all  $q$ .*

*Proof:* Let  $r > n - \dim Z$ , let  $z$  be in  $Z$ , and let  $V \subset Z$  be a relative compact Stein neighborhood of  $z$ , i. e., there is a smooth strictly plurisubharmonic exhaustion function  $\Phi$  of  $V$ . Then  $\Phi \circ f$  is a smooth plurisubharmonic exhaustion function of the relative compact set  $U := f^{-1}(V)$  which is holomorphically convex (using that  $f$  is proper). Since  $f$  is surjective, we obtain  $\sigma(\Phi \circ f) = \sigma(\Phi) = \dim Z$ . So, the assumption gives  $H^r(U, \mathcal{F}) = 0$  since  $r > n - \dim Z$ . Yet, the direct image sheaf  $f_{(r)}(\mathcal{F})$  is the sheaf associated to the presheaf defined by  $V \mapsto H^r(f^{-1}(V), \mathcal{F}) = 0$ . That proves

$$f_{(r)}(\mathcal{F}) = 0.$$

If  $\dim Z = n$ , the Leray spectral sequence (see [Ler50, Chap. II]) implies

$$H^q(X, \mathcal{F}) \cong H^q(Z, f_*\mathcal{F}).$$

Let  $\mathfrak{V} = \{V_i\}_{i \in I}$  be a Leray Covering of  $Z$ , i. e.,

$$H^q(Z, f_*\mathcal{F}) \cong \check{H}^q(\mathfrak{V}, f_*\mathcal{F}). \quad (2.15)$$

Actually, the latter one gives us the topology on the first one. If it is Hausdorff, it is independent of  $\mathfrak{V}$  (see Lem. 4.2 in [Kau67]). For  $\mathfrak{U} := \{f^{-1}(V_i)\}_{i \in I}$ , the definition of the Čech cohomologies implies  $\check{H}^q(\mathfrak{V}, f_*\mathcal{F}) = \check{H}^q(\mathfrak{U}, \mathcal{F})$  with the same topology. Yet, we know that  $\mathfrak{U}$  is already a Leray covering of  $X$ , i. e.,

$$H^q(X, \mathcal{F}) \cong \check{H}^q(\mathfrak{U}, \mathcal{F}) = \check{H}^q(\mathfrak{V}, f_*\mathcal{F}) \cong H^q(Z, f_*\mathcal{F})$$

is topological as well. If  $X$  is regular and  $\mathcal{F}$  locally free, Thm. 2.1 in [Lau67] says that the topologies of the cohomology group given by Leray coverings, differential forms or currents coincide.  $\square$

Let  $f : Y \rightarrow X$  be a holomorphic map between complex spaces, let  $\mathcal{E}$  be a locally free sheaf on  $X$  and let  $\mathcal{F}$  be a coherent analytic sheaf on  $Y$ . Then

$$f_*\mathcal{F} \otimes \mathcal{E} \cong f_*(\mathcal{F} \otimes f^*\mathcal{E}) \quad (2.16)$$

---

<sup>6</sup>Every holomorphically convex space  $X$  is already weakly 1-complete: Using the Remmert Reduction Theorem, we get a Stein space  $Y$  and proper holomorphic map  $\pi : X \rightarrow Y$  (with further properties). Then  $Y$  admits a strictly plurisubharmonic exhaustion function  $\Phi$  (see [Nar62, Thm. II]). Hence,  $\Phi \circ \pi$  is a plurisubharmonic exhaustion function of  $X$ .

which is called the projection formula (cf. Ex. II.5.1 in [Har77]). We obtain the following generalization of the Takegoshi's vanishing theorem.

**Theorem 2.17.** *Let  $X$  be a weakly 1-complete irreducible complex space of dimension  $n$  which is Kähler on relative compact sets, let  $\Phi$  be a smooth plurisubharmonic exhaustion function of  $X$ , and let  $\mathcal{E}$  be a Nakano semi-positive locally free sheaf on  $X$ . Then for each  $q > n - \sigma(\Phi)$ :*

$$H^q(X, \mathcal{E} \otimes \mathcal{K}_X) = 0$$

*if  $H^q(X, \mathcal{E} \otimes \mathcal{K}_X)$  and  $H^{q+1}(X, \mathcal{E} \otimes \mathcal{K}_X)$  are Hausdorff.*

*Proof:* Let  $\pi : M \rightarrow X$  be a resolution of the singularities of  $X$  (cf. [Hir64, Hir77]). Since  $X$  is irreducible,  $M$  is connected. We can assume that  $\pi$  is projective. This implies that  $M$  is Kähler on relative compact open sets (cf. e.g. Lem. 4.4 in [Fuj78]). Since  $\Phi \circ \pi$  is a smooth plurisubharmonic exhaustion function of  $M$  with  $\sigma(\Phi \circ \pi) = \sigma(\Phi)$ , Theorem 2.9 implies: For each  $q > n - \sigma(\Phi)$ ,

$$H^q(M, \pi^* \mathcal{E} \otimes \Omega_M^n) = 0$$

if  $H^{q/q+1}(M, \pi^* \mathcal{E} \otimes \Omega_M^n)$  are Hausdorff. Theorem 2.9 also implies that the assumptions of Theorem 2.14 are satisfied for  $\pi^* \mathcal{E} \otimes \Omega_M^n$  over  $M$  and  $\pi$ , i.e., for each  $q > n - \sigma(\Phi)$ ,

$$H^q(X, \pi_*(\pi^* \mathcal{E} \otimes \Omega_M^n)) \cong H^q(M, \pi^* \mathcal{E} \otimes \Omega_M^n) = 0$$

if  $H^{q/q+1}(X, \pi_*(\pi^* \mathcal{E} \otimes \Omega_M^n))$  are Hausdorff. With the projection formula (2.16), we obtain the claimed.  $\square$

### 3. Vanishing theorems for torsion-free sheaves

In this section, we will prove the main theorems. Let us first recall the definition of Nakano semi-positive coherent analytic sheaves in the sense of H. Grauert and O. Riemenschneider (see [GR70, § 1.2]).

**Def. 3.1.** Let  $\mathcal{S}$  be a coherent analytic sheaf on the complex space  $X$ , let  $S := L(\mathcal{S})$  denote the associated linear space. There exists an open dense set  $X' \subset X_{\text{reg}}$  such that  $S_{X'} = S|_{X'}$  has constant rank, i.e., it is a vector bundle. For each  $x \in X$ , there exist a neighborhood  $U = U(x) \subset X$  and an embedding of  $S_U$  in  $U \times \mathbb{C}^{N(x)}$ . We call  $h = \{h_x\}_{x \in X}$  a (smooth) Hermitian form on  $S$  if all  $h_x$  are Hermitian forms on  $S_x$  and, for an open covering  $\{U_j\}$  of  $X$  with embeddings  $S_{U_j} \subset U_j \times \mathbb{C}^{N_j}$ , there exist smooth Hermitian forms  $h_j$  on  $U_j \times \mathbb{C}^{N_j}$  with  $h_j|_S = h$ . We call  $\mathcal{S}$  Nakano semi-positive if there is a smooth Hermitian form on  $S$  which is Nakano semi-negative on  $S_{X'}$  as a vector bundle.

If  $\mathcal{S}$  is locally free, then the linear space  $L(\mathcal{S})$  is dual to the vector bundle associated to  $\mathcal{S}$ . Hence, in this case, the notations of Nakano semi-positivity coincide. We will only need the following fact: Let  $\mathcal{S}$  be a Nakano semi-positive

sheaf on a complex space, and let  $\pi : Y \rightarrow X$  be a proper modification. Then  $L(\pi^*\mathcal{S}) = \pi^*L(\mathcal{S})$ , and  $L(\pi^T\mathcal{S})$  is embedded in  $L(\pi^*\mathcal{S})$  because of  $\pi^*\mathcal{S} \twoheadrightarrow \pi^T\mathcal{S}$ . With the pull-back on  $L(\pi^*\mathcal{S})$  of the Hermitian metric on  $L(\mathcal{S})$  and the restriction to  $L(\pi^T\mathcal{S})$ , we get that both,  $\pi^*\mathcal{S}$  and  $\pi^T\mathcal{S}$ , are Nakano semi-positive sheaves on  $Y$ .

**3.1. Proof of Theorem I.** Let  $X$  be a weakly 1-complete normal connected complex Kähler space with smooth plurisubharmonic exhaustion function  $\Phi$  and locally free  $\mathcal{K}_X$ . Let  $\mathcal{S}$  be a Nakano semi-positive torsion-free coherent analytic sheaf on  $X$  with normal  $L(\mathcal{S})$  and  $(+)$ , i. e., there is a projective  $\pi : \tilde{X} \rightarrow X$  and a semi-positive locally free analytic sheaf  $\mathcal{L}$  of rank 1 such that  $\pi^T\mathcal{S}$  is locally free and  $\pi^*\mathcal{K}_X = \pi^T\mathcal{K}_X \cong \mathcal{L} \otimes \mathcal{K}_{\tilde{X}}$ .

The locally free sheaf  $\mathcal{E} := \pi^T\mathcal{S} \otimes \mathcal{L}$  is Nakano semi-positive. The composition  $\Phi \circ \pi$  is a plurisubharmonic exhaustion function of  $\tilde{X}$  because  $\pi$  is proper and holomorphic, and  $\sigma(\Phi \circ \pi) = \sigma(\Phi)$ <sup>7</sup> since  $\pi$  is biholomorphic on a dense open set. As  $X$  is Kähler and  $\pi$  is projective, the irreducible complex space  $\tilde{X}$  is Kähler on relative compact open sets (cf. e. g. Lem. 4.4 in [Fuj78]). Theorem 2.17 yields: For each  $q > n - \sigma(\Phi)$ ,

$$H^q(\tilde{X}, \mathcal{E} \otimes \mathcal{K}_{\tilde{X}}) = 0$$

if  $H^q(\tilde{X}, \mathcal{E} \otimes \mathcal{K}_{\tilde{X}})$  and  $H^{q+1}(\tilde{X}, \mathcal{E} \otimes \mathcal{K}_{\tilde{X}})$  are Hausdorff.

Theorem 2.17 also implies that the assumptions of Theorem 2.14 are satisfied for  $\mathcal{E} \otimes \mathcal{K}_{\tilde{X}}$  and  $\pi$ . Therefore, the suitable Hausdorff assumption implies

$$H^q(X, \pi_*(\mathcal{E} \otimes \mathcal{K}_{\tilde{X}})) \cong H^q(\tilde{X}, \mathcal{E} \otimes \mathcal{K}_{\tilde{X}}) = 0 \quad \forall q > n - \sigma(\Phi).$$

Since  $\mathcal{K}_X$  is locally free and  $L(\mathcal{S})$  is normal, we get  $L(\mathcal{S} \otimes \mathcal{K}_X)$  is normal. Therefore, Thm. 1.4 in [RS13] implies

$$\mathcal{S} \otimes \mathcal{K}_X \cong \pi_*(\pi^T\mathcal{S} \otimes \pi^*\mathcal{K}_X) \cong \pi_*(\mathcal{E} \otimes \mathcal{K}_{\tilde{X}}). \quad (3.2)$$

□

**3.2. Proof of Theorem II.** We will now use Theorem I to prove Theorem II with the help of Theorem 2.14. We also need the following fact.

**Lemma 3.3.** *Let  $X$  be a reduced complex space bimeromorphic to a Kähler space and  $U$  a relative compact open set in  $X$ . Then there is a Kähler manifold  $M$  and a proper modification  $g : M \rightarrow U$  of  $U$  such that  $g_{(q)}(\mathcal{F} \otimes \Omega_M^n) = 0$  for all  $q > 0$  and all Nakano semi-positive torsion-free sheaves  $\mathcal{F}$  on  $M$  with  $(+)$  and normal  $L(\mathcal{F})$ .*

*Proof:* By assumption, there exists a Kähler space  $Y$  and a bimeromorphic map  $\alpha : Y \dashrightarrow X$  given by its graph  $\Gamma_\alpha \subset Y \times X$  as analytic set. Let  $\text{pr}_Y : \Gamma_\alpha \rightarrow Y$  and  $\text{pr}_X : \Gamma_\alpha \rightarrow X$  be the holomorphic projections such that  $\alpha = \text{pr}_X \circ \text{pr}_Y^{-1}$ . Hironaka's Chow Lemma (a corollary of the Flattening Theorem, see [Hir75,

<sup>7</sup> Recall  $\sigma(\Phi) := \max_{x \in X_{\text{reg}}} (\text{rk } H(\Phi)_x)$  where  $H(\Phi)_x$  denotes the complex Hessian of  $\Phi$  at  $x$ .

Cor. 2]) gives a projective, particularly, proper bimeromorphic morphism  $\beta : \widetilde{M} \rightarrow Y$  which dominates  $\text{pr}_Y$ , i. e., there is a holomorphic  $h : \widetilde{M} \rightarrow \Gamma_\alpha$  with  $\text{pr}_Y \circ h = \beta$ . We can assume that  $\widetilde{M}$  is smooth by using a resolution of singularities. We obtain the following commutative diagram:

$$\begin{array}{ccccc} \widetilde{M} & \xrightarrow{h} & \Gamma_\alpha & \xrightarrow{\text{pr}_X} & X \\ & \searrow \beta & \downarrow \text{pr}_Y & \nearrow \alpha & \\ & & Y & & \end{array}.$$

Then  $\widetilde{g} := \alpha \circ \beta = \text{pr}_X \circ h$  is a proper modification of  $X$ . Moreover,  $M := \widetilde{g}^{-1}(U)$  is a Kähler manifold – using [Fuj78, Lem. 4.4] for the projective  $\beta : \widetilde{M} \rightarrow Y$  and the relative compact set  $\alpha^{-1}(U)$  in the Kähler space  $Y$  – and  $g := \widetilde{g}|_M : M \rightarrow U$  is a proper modification of  $U$ .

To prove  $g_{(q)}(\mathcal{F} \otimes \Omega_M^n) = 0$ , we will use Theorem I: Let  $x$  be a point in  $U$  and  $W$  an open Stein neighborhood of  $x$  in  $U$ , i. e., there is a smooth strictly plurisubharmonic exhaustion function  $\Psi$  of  $W$ . Since  $g$  is a proper modification, we get a plurisubharmonic exhaustion function  $\Psi \circ g$  of  $g^{-1}(W)$  with  $\sigma(\Psi \circ g) = \sigma(\Psi) = \dim M$ . Hence, the assumptions of Theorem I are satisfied for the holomorphically convex Kähler manifold  $g^{-1}(W)$  and any Nakano semi-positive torsion-free sheaf  $\mathcal{F}$  with  $(+)$  and normal  $L(\mathcal{F})$ . So, we obtain  $H^q(g^{-1}(W), \mathcal{F} \otimes \Omega_M^n) = 0$  for all  $q > \dim X - \dim M = 0$  and, finally,  $g_{(q)}(\mathcal{F} \otimes \Omega_M^n) = 0$ .  $\square$

*Proof of Theorem II:* Let  $X$  be a normal complex space of pure dimension  $n$  with locally free  $\mathcal{K}_X$  which is bimeromorphic to a Kähler space, let  $\mathcal{S}$  be a (semi-globally) Nakano semi-positive torsion-free coherent analytic sheaf on  $X$  with  $(+)_\text{loc}$  and normal  $L(\mathcal{S})$ , and let  $f : X \rightarrow Z$  be a proper surjective holomorphic map to a complex space  $Z$ . To prove the vanishing of the higher direct images of  $\mathcal{S} \otimes \mathcal{K}_X$ , we have to check that the assumptions of Theorem 2.14 are satisfied, i. e., if a relative compact open set  $U \subset X$  possesses a smooth plurisubharmonic exhaustion function  $\Phi$ , then  $H^r(U, \mathcal{S} \otimes \mathcal{K}_X) = 0$  for  $r > n - \sigma(\Phi)$ .

Let  $U \subset X$  be a relative compact open set with smooth plurisubharmonic exhaustion function  $\Phi$ . By assumption,  $\mathcal{S}_U$  satisfies  $(+)$ , i. e., there is a proper modification  $\pi : \widetilde{U} \rightarrow U$  and a semi-positive locally free sheaf  $\mathcal{L}$  on  $\widetilde{U}$  of rank 1 such that  $\pi^T \mathcal{S}_U$  is locally free and  $\pi^* \mathcal{K}_U \cong \mathcal{L} \otimes \mathcal{K}_{\widetilde{U}}$ . In particular, the sheaf  $\mathcal{E} := \pi^T \mathcal{S}_U \otimes \mathcal{L}$  is locally free and Nakano semi-positive. Since  $L(\mathcal{S}_U \otimes \mathcal{K}_U)$  is normal, Thm. 1.4 in [RS13] implies

$$\mathcal{S}_U \otimes \mathcal{K}_U \cong \pi_*(\mathcal{E} \otimes \mathcal{K}_{\widetilde{U}}). \quad (3.4)$$

Since  $\widetilde{U}$  is bimeromorphic to  $U$ , it is bimeromorphic to a Kähler space, as well. Therefore, Lemma 3.3 gives a Kähler manifold  $M$  and a proper modification  $g : M \rightarrow \widetilde{U}$  with  $g_{(q)}(\mathcal{F} \otimes \Omega_M^n) = 0$  for  $q \geq 1$  and  $\mathcal{F} := g^* \mathcal{E}$ .

For all holomorphically convex open  $V \subset \tilde{U}$  with smooth plurisubharmonic exhaustion function  $\Psi$  and  $W := g^{-1}(V)$ , we get

$$\begin{aligned} 0 &\stackrel{\text{Thm. I}}{=} H^r(W, \mathcal{F} \otimes \Omega_W^n) \stackrel{\text{Leray}}{\cong} H^r(V, g_*(g^*\mathcal{E} \otimes \Omega_W^n)) \\ &\stackrel{(2.16)}{\cong} H^r(V, \mathcal{E} \otimes \mathcal{K}_V) \quad \forall r > n - \sigma(\Psi), \end{aligned} \quad (3.5)$$

i.e., the assumptions of Theorem 2.14 holds for  $\mathcal{E} \otimes \mathcal{K}_{\tilde{U}}$  and  $\pi$ . Therefore, Theorem 2.14 and (3.4) imply

$$H^r(\tilde{U}, \mathcal{E} \otimes \mathcal{K}_{\tilde{U}}) \cong H^r(U, \pi_*(\mathcal{E} \otimes \mathcal{K}_{\tilde{U}})) \cong H^r(U, \mathcal{S}_U \otimes \mathcal{K}_U). \quad (3.6)$$

Using (3.5) for  $V = \tilde{U}$  and  $\Psi = \Phi$ , we obtain

$$H^r(U, \mathcal{S}_U \otimes \mathcal{K}_U) = 0 \quad \forall r > n - \sigma(\Phi). \quad \square$$

In the proof, the Nakano semi-positivity of  $\mathcal{S}$  is just needed on preimages of small Stein sets in  $Z$  under  $f$  / on relative compact weakly 1-complete subsets of  $X$  (cf. Def. of  $(+)_{\text{loc}}$ ).

#### 4. Modifications of coherent analytic sheaves

In [RS13], J. Ruppenthal and the author studied the behavior of the direct and inverse images of coherent analytic sheaves under proper modifications. The results presented in this section are interesting in this context, but moreover, they give us some kind of necessity of the normality condition on the linear space of the sheaf in Theorem I.

First, let us present an alternative result to Thm. 1.4 in [RS13] where the normality assumption is not needed anymore to show that the composition of the direct image and preimage functor is an isomorphism:

**Theorem 4.1.** *Let  $X$  be an irreducible Cohen-Macaulay space, and let  $\mathcal{S}$  be a coherent analytic sheaf of rank 1 on  $X$  such that, for each  $p \in X$ , there exist a neighborhood  $U$  and a free resolution  $\mathcal{O}_U^m \rightarrow \mathcal{O}_U^{m+1} \rightarrow \mathcal{S}_U \rightarrow 0$  (i.e., the homological dimension of  $\mathcal{S}$  is at most 1) and the singular locus of  $\mathcal{S}$  is at least  $(m+1)$ -codimensional, or such that (more weakly) the linear space  $L(\mathcal{S})$  associated to  $\mathcal{S}$  is Cohen-Macaulay and irreducible. Let  $\varphi = \varphi_{\mathcal{S}}: Y \rightarrow X$  denote the monoidal transformation of  $X$  with respect to  $\mathcal{S}$ . Then the canonical morphism  $\mathcal{S} \rightarrow \varphi_*\varphi^*\mathcal{S}$  induces an isomorphism*

$$\mathcal{S} \xrightarrow{\sim} \varphi_*\varphi^T\mathcal{S}.$$

For  $m \leq 2$ , the assumption on the resolution of  $\mathcal{S}_U$  in Theorem 4.1 is always satisfied if  $X$  is factorial and Cohen-Macaulay and  $\mathcal{S}$  is torsion-free and generated by  $\text{rk } \mathcal{S} + m$  elements (see Thm. 1.3 in [RS13]). We will use the following fact.

**Lemma 4.2.** *Let  $S \subset U \times \mathbb{C}^N$  be a linear space on an irreducible Cohen-Macaulay space  $U$  of rank  $r$  defined by  $m$  holomorphic fiber-wise linear functions  $h_1, \dots, h_m$  such that  $N=r+m$  and the singular locus of  $S$  is at least  $m$ -codimensional in  $U$ . Then  $S$  is Cohen-Macaulay. If the codimension of the singular locus is at least  $m+1$ , then  $S$  is irreducible. If  $U$  is a locally complete intersection, then  $S$  is a locally complete intersection, as well.*

*Proof:* Since  $S = \{h_1 = \dots = h_m = 0\}$ , we get  $\text{codim}_{(p,z)} S \leq m$  for all  $(p, z) \in S$ . Let  $A \subset U$  denote the singular locus of  $S$ , i.e., the set where  $S$  is not locally free, and let  $E$  denote the primary component of  $S$ , i.e., the irreducible component of  $S$  containing  $S_{U \setminus A}$ . Then  $\dim E = \dim U + r$ , i.e.,  $\text{codim} E = m$ . We set  $T := (A \times \mathbb{C}^N) \cap S$ . By the assumption and  $T \subset A \times \mathbb{C}^N$ ,  $\text{codim} T \geq \text{codim}(A \times \mathbb{C}^N) \geq m$ . Hence,  $\text{codim}_{(p,z)} S = m$  for all points  $(p, z) \in S$ . Since  $\mathcal{O}_S = \mathcal{O}_{U \times \mathbb{C}^N} / (h_1, \dots, h_m)$ , we get  $S$  is Cohen-Macaulay (see e.g. Prop. 5.2 in [PR94]). Additionally,  $S$  is a locally complete intersection if  $U$  is a locally complete intersection.

Let us assume that  $S$  is not irreducible, i.e.,  $T \setminus E \neq \emptyset$ . For all  $(p, z) \in T \setminus E$ , there is a neighborhood  $V$  of  $(p, z)$  such that  $T \cap V = \{h_1 = \dots = h_m = 0\}$ , i.e.,  $\text{codim}_{(p,z)} T \leq m$ . We get  $\text{codim}_U A \leq \text{codim}_{U \times \mathbb{C}^N} T \leq m$ . That proves the second claim.  $\square$

*Proof of Theorem 4.1:* Let  $S$  denote the linear space associated to  $\mathcal{S}$ . With the lemma from above or by the assumption, we get  $S$  is Cohen-Macaulay and irreducible (in particular,  $\mathcal{S}$  is torsion-free, see [RS13, Lem. 3.2]). Let  $E := L(\varphi^T \mathcal{S}) \subset \varphi^* S = L(\varphi^* \mathcal{S})$  denote the linear space associated to the torsion-free preimage of  $\mathcal{S}$  and  $\text{pr} : E \rightarrow S$  be the restriction of the projection of the fiber product  $Y \times_X S = \varphi^* S$  to  $S$ . Then  $E$  is irreducible and the proper mapping theorem implies  $\text{pr}(E) = S$ , i.e.,  $\text{pr}$  is a proper modification of  $S$ .

The biholomorphism between  $\mathbb{C}^{m+1} \setminus 0$  and the universal line bundle without zero section  $\mathcal{O}_{\mathbb{CP}^m}(1) \setminus (\mathbb{CP}^m \times 0)$  defined by  $z \mapsto ([z], z)$  induces a biholomorphic map from  $S_p \setminus 0 \xrightarrow{\sim} E_{\varphi^{-1}(p)} \setminus (\varphi^{-1}(p) \times 0)$  for each  $p \in X$ , which is the inverse map of

$$\text{pr} : E \setminus (Y \times 0) \rightarrow S \setminus (X \times 0)$$

(cf. the construction of  $\varphi = \varphi_{\mathcal{S}}$  in [Rie71, § 2]).

Let  $A$  denote the singular locus of  $\mathcal{S}$ , i.e., the set where  $\mathcal{S}$  is not locally free (and  $S$  is not a line bundle), and set  $B := \varphi^{-1}(A)$ . Since  $\varphi$  is biholomorphic outside of  $B$ ,  $\text{pr} = (\varphi, \text{id}_{\mathbb{C}^{m+1}})|_E$  is already a biholomorphic map on the complement of  $B$  and  $A$ :

$$\text{pr} : E \setminus (B \times 0) \xrightarrow{\sim} S \setminus (A \times 0). \quad (4.3)$$

Since  $A \times 0$  is at least of codimension 2 in  $S$  and  $S$  is Cohen-Macaulay, every holomorphic function on  $S \setminus (A \times 0)$  extends to  $S$  (see Cor. 5.9 in [PR94]). Hence, for all open sets  $U \subset X$ , we get

$$\begin{aligned} (\varphi_* \varphi^T \mathcal{S})(U) &\stackrel{\text{def}}{=} (\varphi^T \mathcal{S})(\varphi^{-1}(U)) \cong \text{Hom}(E_{\varphi^{-1}(U)}, \varphi^{-1}(U) \times \mathbb{C}) \\ &\cong \text{Hom}(S_U, U \times \mathbb{C}) \cong \mathcal{S}(U), \end{aligned}$$

where the second and the last isomorphism are given by the construction of the linear spaces associated to  $\varphi^T \mathcal{S}$  and  $\mathcal{S}$ , resp.  $\square$

**Remark 4.4.** For the irreducible Cohen-Macaulay space  $S$  of rank 1, we get that  $S$  is normal if and only if  $S \setminus (A \times 0)$  is normal. Since  $E \setminus (B \times 0) \xrightarrow{\sim} S \setminus (A \times 0)$  (4.3), and since  $E$  is vector bundle, this is furthermore equivalent to  $Y$  being normal.

For the torsion-free inverse image of the direct image sheaf under a 1:1 modification, we obtain:

**Theorem 4.5.** *Let  $X$  be a locally irreducible complex space, let  $\psi : \widehat{X} \rightarrow X$  be the normalization of  $X$ , and let  $\mathcal{F}$  be a torsion-free coherent analytic sheaf on  $\widehat{X}$ . Then the canonical morphism  $\psi^* \psi_* \mathcal{F} \rightarrow \mathcal{F}$  induces an isomorphism*

$$\psi^T \psi_* \mathcal{F} \cong \mathcal{F}.$$

*Proof:* Since the 1-sheeted covering  $\psi$  is a homeomorphism ( $X$  is locally irreducible), we get, for all  $q \in \widehat{X}$ ,

$$\mathcal{F}_q = (\psi_* \mathcal{F})_{\psi(q)} \stackrel{\text{def}}{=} (\psi^{-1} \psi_* \mathcal{F})_q.$$

By the definition of the (analytic) inverse image sheaf, we obtain

$$\psi^* \psi_* \mathcal{F} \stackrel{\text{def}}{=} \psi^{-1} \psi_* \mathcal{F} \otimes_{\psi^{-1} \mathcal{O}_X} \mathcal{O}_{\widehat{X}} = \mathcal{F} \otimes_{\psi^{-1} \mathcal{O}_X} \mathcal{O}_{\widehat{X}}.$$

Yet, the injective map  $\psi^{-1} \mathcal{O}_X \hookrightarrow \mathcal{O}_{\widehat{X}}$  gives us the surjectivity of the canonical morphism:

$$\psi^* \psi_* \mathcal{F} = \mathcal{F} \otimes_{\psi^{-1} \mathcal{O}_X} \mathcal{O}_{\widehat{X}} \twoheadrightarrow \mathcal{F} \otimes_{\mathcal{O}_{\widehat{X}}} \mathcal{O}_{\widehat{X}} = \mathcal{F}, s \otimes r \mapsto r \cdot s.$$

On the other hand, it is not hard to see that the canonical morphism induces an injective map (see Lem. 5.1 in [RS13]):

$$\psi^T \psi_* \mathcal{F} \hookrightarrow \mathcal{F}, s \otimes r \mapsto \mathcal{I}(\psi^* \psi_* \mathcal{F}) \mapsto r \cdot s.$$

$\square$

**Remark 4.6.** Let  $\mathcal{S}$  be a coherent analytic sheaf of rank 1 on a complex manifold  $M$  such that the linear space  $S = L(\mathcal{S})$  is Cohen-Macaulay and irreducible, but not normal, and let  $\varphi : Y \rightarrow M$  be the monoidal transformation of  $M$  with respect to  $\mathcal{S}$ . We obtain with Theorem 4.1:

$$\mathcal{S} \otimes \mathcal{K}_M \cong \varphi_*(\varphi^T \mathcal{S} \otimes \varphi^* \mathcal{K}_M).$$

The sheaf  $\mathcal{E} := \varphi^T \mathcal{S} \otimes \varphi^* \mathcal{K}_M$  is locally free. For our purpose of generalizing Takegoshi's vanishing theorem, we need a proper modification  $\pi : Z \rightarrow Y$  and a locally free sheaf  $\widetilde{\mathcal{E}}$  with  $\mathcal{E} \cong \pi_*(\widetilde{\mathcal{E}} \otimes \mathcal{K}_Z)$ . With the projection formula for locally free sheaves and a normalization, we can assume that  $Z$  is normal. Hence, we can apply the following theorem which gives then a contradiction to the assumption that  $S$  and, hence, also  $Y$  are not normal (see Remark 4.4). This means that the assumption on normality of  $L(\mathcal{S})$  is necessary for a generalization of Takegoshi's vanishing theorem by use of a monoidal transformation.



**Theorem 4.7.** *If  $\mathcal{E}$  is a locally free sheaf with positive rank on a locally irreducible complex space  $Y$  such that there exist a proper modification  $\pi : Z \rightarrow Y$  with normal  $Z$  and a coherent analytic sheaf  $\mathcal{F}$  with  $\pi_*\mathcal{F} \cong \mathcal{E}$ , then  $Y$  is normal.*

*Proof:* Let  $\psi : \widehat{Y} \rightarrow Y$  be a normalization of  $Y$ . Since  $Z$  is normal,  $\pi$  factorizes over the normalization, i. e.,  $\exists \widehat{\pi} : Z \rightarrow \widehat{Y}$  with  $\pi = \psi \circ \widehat{\pi}$ . Therefore,  $\mathcal{E} \cong \psi_*\widehat{\pi}_*\mathcal{F}$ . For  $\widehat{\mathcal{F}} := \widehat{\pi}_*\mathcal{F}$ , Theorem 4.5 implies

$$\mathcal{E} \cong \psi_*\widehat{\mathcal{F}} \cong \psi_*\psi^T\psi_*\widehat{\mathcal{F}} \cong \psi_*\psi^T\mathcal{E} = \psi_*(\psi^*\mathcal{E} \otimes \mathcal{O}_{\widehat{Y}}) \stackrel{(2.16)}{\cong} \mathcal{E} \otimes \psi_*\mathcal{O}_{\widehat{Y}} \cong \mathcal{E} \otimes \widehat{\mathcal{O}}_Y.$$

Since  $\mathcal{E}$  is locally free of positive rank, we obtain  $\widehat{\mathcal{O}}_Y \cong \mathcal{O}_Y$ , i. e.,  $Y$  is normal.  $\square$

We immediately obtain

**Corollary 4.8.** *The Grauert-Riemenschneider canonical sheaf on a non-normal locally irreducible complex space is not locally free.*

## 5. Submanifolds of holomorphically convex manifolds

In this section, we give an example of a torsion-free coherent analytic non-locally-free sheaf which satisfies (+).

Let  $M$  be a complex manifold of dimension  $n$ , let  $Y$  be a (connected) submanifold of  $M$  of codimension  $m$ , and let  $\mathcal{J} = \mathcal{J}_Y$  be the (reduced) ideal sheaf of  $Y$ . If  $m > 1$ , then  $\mathcal{J}$  is not locally free. The monoidal transformation with respect to  $\mathcal{J}$  of  $M$  is given by the blow up  $\varphi : \widetilde{M} \rightarrow M$  of  $M$  with center in  $Y$  such that  $\varphi^T\mathcal{J}$  is locally free. Let  $Z = \varphi^{-1}(Y)$  denote the exceptional divisor/set of  $\varphi$ , and let  $\mathcal{O}(-Z)$  denote the ideal sheaf on  $\widetilde{M}$  of holomorphic functions vanishing on  $Z$ . In Sect. 7 of [RS13], it has been proven that

$$\varphi^T\mathcal{J} = \mathcal{O}(-Z) \text{ and } \mathcal{J} \cong \varphi_*\mathcal{O}(-Z). \quad (5.1)$$

Hence,  $\mathcal{J} \cong \varphi_*\varphi^T\mathcal{J}$ , which is already the statement of Thm. 1.4 in [RS13]. Therefore, one need not verify the normality of  $L(\mathcal{J})$  to prove Theorem I for  $\mathcal{J}$ : one can use the second isomorphism of (5.1) combined with the projection formula to get (3.2).

On the other hand, for  $m = 2$ ,  $L(\mathcal{J})$  is a hypersurface. So, one can prove the normality of  $L(\mathcal{J})$  easily by computing the codimension of the singular set without using (5.1).

For the canonical sheaf on  $\widetilde{M}$ , we have (see e. g. Prop. VII.12.7 in [Dem12])

$$\Omega_{\widetilde{M}}^n = \varphi^*\Omega_M^n \otimes \mathcal{O}((m-1)Z).$$

Combining this with (5.1), we get

$$\varphi^T(\mathcal{J} \otimes \Omega_M^n) = \Omega_{\widetilde{M}}^n \otimes \mathcal{O}(-mZ).$$

Under the assumption that  $\mathcal{J}$  is semi-positive (e. g.  $Y$  is the zero set of finitely many globally defined holomorphic functions), we get that  $\varphi^T\mathcal{J} \cong \mathcal{O}(-Z)$  is

semi-positive, as well. Let  $L$  denote the line bundle on  $\widetilde{M}$  associated to  $\mathcal{O}(-Z)$ , such that  $L^{\otimes k}$  is the line bundle associated to  $\mathcal{O}(-kZ)$ . Since  $\Theta(L^{\otimes k}) = k\Theta(L)$ , semi-positivity of  $\varphi^T \mathcal{J} = \mathcal{O}(-Z)$  gives us the semi-positivity of  $\mathcal{O}(-(m-1)Z)$ . Hence,  $\mathcal{J}$  satisfies (+) (with  $\mathcal{L} = \mathcal{O}(-(m-1)Z)$ ), and it is derived from the semi-positivity of  $\mathcal{J}$ . Applying Theorem I, we get

**Corollary 5.2.** *Let  $M$  be a holomorphically convex Kähler manifold of dimension  $n$ , let  $\Phi$  be a smooth plurisubharmonic exhaustion function of  $M$ , let  $\mathcal{E}$  be a Nakano semi-positive locally free analytic sheaf on  $M$ , and let  $\mathcal{J}$  be a semi-positive ideal sheaf<sup>8</sup> given by a submanifold of  $M$ . Then for each  $q > n - \sigma(\Phi)$ :*

$$H^q(X, \mathcal{J} \otimes \mathcal{E} \otimes \Omega_M^n) = 0.$$

Further, we obtain a vanishing result for submanifolds of weakly 1-complete manifolds: The short exact sequence

$$0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_M/\mathcal{J} \rightarrow 0$$

gives the short exact sequence

$$0 \rightarrow \mathcal{J} \otimes \Omega_M^n \rightarrow \Omega_M^n \rightarrow \Omega_M^n \otimes \mathcal{O}_M/\mathcal{J} \rightarrow 0. \quad (5.3)$$

Since

$$(\Omega_M^n \otimes \mathcal{O}_M/\mathcal{J})|_Y = \Omega_M^n|_Y \otimes \mathcal{O}_Y$$

and

$$\Omega_Y^{n-m} = \Omega_M^n|_Y \otimes \det \mathcal{N}_{Y/M}$$

(adjunction formula, see e.g. (5.26a) in [PR94]; where  $\mathcal{N}_{Y/M}$  denotes the sheaf of sections of the normal bundle of  $Y$ ), the long exact sequence of cohomology associated to (5.3) implies

**Corollary 5.4.** *Let  $M$  be a holomorphically convex Kähler manifold of dimension  $n$ , let  $\Phi$  be a smooth plurisubharmonic exhaustion function of  $M$ , and let  $Y$  be a submanifold of  $M$  with semi-positive ideal sheaf and of dimension  $r$ . Then for each  $q > n - \sigma(\Phi)$ :*

$$H^q(Y, \Omega_Y^r \otimes \det \mathcal{N}_{Y/M}^*) = 0.$$

*In the case that the normal bundle of  $Y$  (or the determinant of it) is the restriction of a Nakano semi-positive vector bundle, we get for each  $q > n - \sigma(\Phi)$ :*

$$H^q(Y, \Omega_Y^r) = 0.$$

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<sup>8</sup>E.g. generated by finitely many globally defined holomorphic functions.

## 6. Sheaves with torsion

Let  $X$  be a holomorphically convex normal Kähler space of dimension  $n$ ,  $\Phi$  a smooth plurisubharmonic exhaustion function of  $X$ , and let  $\mathcal{S}$  be a Nakano semi-positive coherent analytic sheaf on  $X$  satisfying (+). We define  $\mathcal{T} := \mathcal{T}(\mathcal{S})$  as the torsion sheaf of  $\mathcal{S}$  and obtain the exact sequence

$$0 \rightarrow \mathcal{T} \rightarrow \mathcal{S} \rightarrow \mathcal{S}/\mathcal{T} \rightarrow 0.$$

Assuming that the Grauert-Riemenschneider canonical sheaf  $\mathcal{K}_X$  is locally-free, we get the exact sequence

$$0 \rightarrow \mathcal{T} \otimes \mathcal{K}_X \rightarrow \mathcal{S} \otimes \mathcal{K}_X \rightarrow (\mathcal{S}/\mathcal{T}) \otimes \mathcal{K}_X \rightarrow 0.$$

This yields the long exact sequence of cohomology:

$$\begin{aligned} 0 \longrightarrow (\mathcal{T} \otimes \mathcal{K}_X)(X) \longrightarrow (\mathcal{S} \otimes \mathcal{K}_X)(X) \longrightarrow ((\mathcal{S}/\mathcal{T}) \otimes \mathcal{K}_X)(X) \longrightarrow \dots \\ \dots \rightarrow H^q(X, \mathcal{T} \otimes \mathcal{K}_X) \rightarrow H^q(X, \mathcal{S} \otimes \mathcal{K}_X) \rightarrow H^q(X, (\mathcal{S}/\mathcal{T}) \otimes \mathcal{K}_X) \rightarrow \dots \end{aligned}$$

Since the restriction of the Hermitian metric on  $L(\mathcal{S})$  gives a Hermitian metric on the embedded space  $L(\mathcal{S}/\mathcal{T})$ , the torsion-free coherent analytic sheaf  $\mathcal{S}/\mathcal{T}$  is Nakano semi-positive.  $\mathcal{S}$  passes (+) on to  $\mathcal{S}/\mathcal{T}$  because of  $\pi^T(\mathcal{S}) = \pi^T(\mathcal{S}/\mathcal{T})$ . Assuming that  $L(\mathcal{S}/\mathcal{T})$  is normal, we obtain  $H^q(X, (\mathcal{S}/\mathcal{T}) \otimes \mathcal{K}_X) = 0$  for all  $q > n - \sigma(\Phi)$  by Theorem I. Thus, the long exact sequence gives isomorphisms

$$H^q(X, \mathcal{T} \otimes \mathcal{K}_X) \cong H^q(X, \mathcal{S} \otimes \mathcal{K}_X) \quad \forall q > n - \sigma(\Phi) + 1$$

and the surjective homomorphism

$$H^{n-\sigma(\Phi)+1}(X, \mathcal{T} \otimes \mathcal{K}_X) \twoheadrightarrow H^{n-\sigma(\Phi)+1}(X, \mathcal{S} \otimes \mathcal{K}_X).$$

On the other hand,  $\mathcal{T}$  and, hence,  $\mathcal{T} \otimes \mathcal{K}_X$  have support on an analytic set  $A \subset X$  with  $r := \dim A = \sup_{x \in A} \dim_x A < n$ . Let  $\iota : A \hookrightarrow X$  denote the embedding. As  $\mathcal{T} \otimes \mathcal{K}_X$  is a coherent analytic sheaf with support in  $A$ , we have

$$\mathcal{T} \otimes \mathcal{K}_X = \iota_* \iota^* (\mathcal{T} \otimes \mathcal{K}_X). \quad (6.1)$$

This is easy to see by working in the category of linear (fiber) spaces associated to coherent analytic sheaves: For linear spaces,  $\iota^*$  means nothing else but restriction of the linear space to the subvariety  $A$ , and  $\iota_*$  means just trivial extension of the space over  $A$  to  $X$ . Note that (6.1) is not true for sheaves which are not coherent.

We get (cf. e.g. Prop. 5.2 in [Ive84, Chap. II])

$$H^q(A, \iota^* (\mathcal{T} \otimes \mathcal{K}_X)) \cong H^q(X, \iota_* \iota^* (\mathcal{T} \otimes \mathcal{K}_X)) \stackrel{(6.1)}{=} H^q(X, \mathcal{T} \otimes \mathcal{K}_X).$$

Using  $H^q(A, \iota^* (\mathcal{T} \otimes \mathcal{K}_X)) = 0$  for  $q > r$  (see e.g. Thm. 10.2 in [Ive84, Chap. II]), we conclude:

**Theorem 6.2.** *Let  $X$  be a holomorphically convex normal connected Kähler space of dimension  $n$  such that  $\mathcal{K}_X$  is locally free, let  $\Phi$  denote a smooth plurisubharmonic exhaustion function of  $X$ , and let  $\mathcal{S}$  be a Nakano semi-positive sheaf*

on  $X$  with (+) and normal  $L(\mathcal{S}/\mathcal{T}(\mathcal{S}))$ . Then we get, for  $q > \max\{n - \sigma(\Phi), \dim \operatorname{supp} \mathcal{T}(\mathcal{S})\}$ ,

$$H^q(X, \mathcal{S} \otimes \mathcal{K}_X) = 0.$$

We give a counterexample to show that this result is sharp (with respect to the dimension). Let  $M$  be a holomorphically convex Kähler manifold of dimension  $n$  which is not Stein and admits a smooth plurisubharmonic exhaustion function which is in (at least) one point strictly plurisubharmonic (consider e. g. the blow up of  $\mathbb{C}^n$  in a point). Let  $A$  be a compact analytic subset of  $X$  and  $\iota : A \hookrightarrow M$  the embedding of  $A$ . For any  $0 < q \leq \dim A$ , one can find such spaces  $M$  and  $A$  admitting a coherent analytic sheaf  $\mathcal{F}$  on  $A$  such that  $H^q(A, \mathcal{F}) \neq 0$ . We set

$$\mathcal{S} := \iota_* \mathcal{F} \otimes (\Omega_M^n)^*.$$

$\mathcal{S}$  is Nakano semi-positive as it vanishes outside a thin set,  $L(\mathcal{S}/\mathcal{T}(\mathcal{S})) = M \times 0$  is normal, and  $\operatorname{id}_M^T \mathcal{S} = \mathcal{S}/\mathcal{T}(\mathcal{S}) = 0$ , hence,  $\mathcal{S}$  satisfies (+). Yet, we have

$$H^q(M, \mathcal{S} \otimes \Omega_M^n) = H^q(M, \iota_* \mathcal{F}) \cong H^q(A, \mathcal{F}) \neq 0.$$

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